International Journal of Theoretical Physics, Vol. 42, No. 7, July 2003 (© 2003)

States on Subspaces of Inner Product Spaces with the Gleason Property

Anatolij Dvurečenskij[1]

Received February 17, 2003

We show that the range of every finitely additive state on the system $\mathcal{F}(E)$ of all orthogonally closed subspaces of an infinite-dimensional inner product space E satisfying the Gleason property is equal to the real interval [0, 1]. Every pre-Hilbert space satisfies the Gleason property, and in Keller spaces it fails to hold.

KEY WORDS: inner product space; orthogonally closed subspace; finitely additive state; range of a state; Hilbert space; Keller space.

1. INTRODUCTION

The system $\mathcal{L}(H)$ of all closed subspaces of a Hilbert space, H, is the most important example of quantum logics which plays a fundamental role in the axiomatization of quantum mechanics (see, e.g., Varadarajan, 1968). The fundamental result of Gleason (1957) says that there is a one-to-one correspondence among σ-additive states s, on $\mathcal{L}(H)$, $3 \leq \dim H \leq \aleph_0$, and trace operators T, on H given by

$$s(M) = \mathrm{tr}(T P_M), \quad M \in \mathcal{L}(H), \tag{1.1}$$

where P_M is the orthogonal projector from H onto M.

The range of probability measures was probably firstly systematically studied by Liapounoff (see e.g., Armstrong and Prikry, 1981), where some conditions are given which guarantee that the range of the probability measure equal the real interval [0, 1].

Recently, it was shown that the range of every finitely additive state on the system of all orthogonally closed subspaces of an infinite-dimensional pre-Hilbert space S is also equal to the whole real interval [0, 1]. (Dvurečenskij and Pták, 2002) For the finite-dimensional case of S this is not true, in general.

[1] Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, SK-814 73 Bratislava, Slovakia; e-mail: dvurecen@mat.savba.sk.

In the present paper, we study the range of finitely additive states on the system of all orthogonally closed subspaces $\mathcal{F}(E)$ of an infinite-dimensional inner product space E. We show that if such an E satisfies the Gleason property, then the range of any finitely additive state is equal to the whole real interval [0, 1]. We show that every infinite-dimensional pre-Hilbert space satisfies the Gleason property, but in any infinite-dimensional nonclassical Keller space it fails to hold, and the range of a state can be even a finite set.

2. ORTHOGONALLY CLOSED SUBSPACES OF INNER PRODUCT SPACES

Let K be a division ring with char $K \neq 2$ and with an involution *: $K \to K$ such that $(\alpha+\beta)^* = \alpha^* + \beta^*$, $(\alpha\beta)^{**} = \beta^*\alpha^*$, $\alpha^{**} = \alpha$ for all $\alpha, \beta \in K$. Let E be a (left) vector space over K equipped with a Hermitian form $(\cdot,\cdot)$: $E \times E \to K$, i.e. $(\cdot,\cdot)$ satisfies, for all $x, y, z \in E$ and all $\alpha, \beta \in K$, $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$, $(x, \alpha y + \beta z) = (x, y)\alpha^* + (x, z)\beta^*$, $(x, y) = (y, x)^*$. The triplet $(E, K, (\cdot,\cdot))$ is said to be an *inner product space (a generalized inner product space)* or *a quadratic space* if $(x, x) = 0$ implies $x = 0$, and unless confusion threatens, we usually refer to E rather than to $(E, K, (\cdot,\cdot))$.

For any nonzero vector $x \in E$, let sp(x) be a one-dimensional subspace of E generated by the vector x. Two vectors $x, y \in E$ are orthogonal, and we write $x \perp y$, if $(x, y) = 0$.

For any subset $M \subseteq E$, we put $M^\perp = \{x \in E: (x, y) = 0 \text{ for any } y \in M\}$. Let $\mathcal{F}(E)$ denote the family of all orthogonally closed subspaces of E, i.e.,

$$\mathcal{F}(E) = \{M \subseteq E: M^{\perp\perp} = M\}, \tag{2.1}$$

and let $\mathcal{E}(E)$ denote the set of all splitting subspaces of E, i.e.,

$$\varepsilon(E) = \{M \subseteq E: M^\perp + M = E\}. \tag{2.2}$$

Then $M \in \mathcal{E}(E)$ whenever dim $M < \infty$ or dim $M^\perp < \infty$, in particular $\{0\}$, $E \in \mathcal{E}(E)$, and[2]

$$\mathcal{E}(E) \subseteq \mathcal{F}(E).$$

E is said to be *orthomodular* iff $\mathcal{F}(E) \subseteq \mathcal{E}(E)$. Any finite-dimensional inner product space E is orthomodular. If E is a real, complex, or quaternion inner product space, then E is orthomudular iff E is a Hilbert space. Keller (1980) gave examples of nonclassical infinite-dimensional inner product spaces which are orthomodular.

[2] Indeed, let $M + M^\perp = E$ and since $M \subseteq M^{\perp\perp}$, it suffices to prove that $M^{\perp\perp} \subseteq M$. We first note that $\{0\} = E^\perp = (M + M^\perp)^\perp = M^\perp \cap M^{\perp\perp}$. Let $x \in M^{\perp\perp}$. Then there exist $x_1 \in M$ and $x_2 \in M^\perp$ such that $x = x_1 + x_2$. Hence $x_2 = x - x_1 \in M^{\perp\perp}$ and therefore $x_2 = 0$ and, consequently, $x \in M$.

In general, the system $\mathcal{F}(E)$ is an orthocomplemented complete lattice while $\varepsilon(E)$ is an orthomodular poset, i.e., if $M, N \in \varepsilon(E)$, $M \perp N$, then $M \vee N \in \varepsilon(E)$, and in this case $M \vee N = M + N$, and if $M \subseteq N$, then the orthomodular law

$$N = M \vee (N \cap M^{\perp} \tag{2.3}$$

holds.

Moreover, according to Maeda and Maeda (1970, Lemma 33.3),

(i) if $M \in \mathcal{F}(E)$ and $x \in E$ is a nonzero vector, then $M \vee \operatorname{sp}(x) = M + \operatorname{sp}(x) \in \mathcal{F}(E)$,
(ii) $\bigwedge M_i = \bigcap_i M_i$ for any system $\{M_i\}$ from $\mathcal{F}(E)$.

A mapping $s : \mathcal{F}(E) \to [0, 1]$ such that (i) $s(E) = 1$ and (ii) $s(M \vee N) = s(M) + s(N)$ whenever M and N are mutually orthogonally elements from $\mathcal{F}(E)$ is said to be a *finitely additive state* (or a *state* in the abbreviation). If (ii) is changed to (ii)* $s(\bigvee_{i=1}^{\infty} M_i) = \sum_{i=1}^{\infty} s(M_i)$ for any sequence of mutually orthogonal subspaces $\{M_i\}$ from $\mathcal{F}(E)$, s is said to be a *σ-additive state*.

For example, if E is a finite-dimensional inner product space, then the mapping $s : \mathcal{F}(E) \to [0, 1]$ defined by

$$s(M) = \dim(M)/\dim(E), \quad M \in \mathcal{F}(E), \tag{2.4}$$

is a finitely additive state.

Let s be a finitely additive state on $\mathcal{F}(E)$. Let us set $\operatorname{Ran}(s) := \{s(M)\colon M \in \mathcal{F}(S)\}$. For any integer $i = 0, 1, \ldots,$ let us define $\operatorname{Ran}(s)_i := \{s(M)\colon \in \mathcal{F}(E), \dim M = i\}$.

If E is a finite-dimensional inner product space, and s is defined by (2.4), then $\operatorname{Ran}(s) = \{0, 1/n, 2/n, \ldots, n/n\}$, where $n = \dim E$.

If E is an infinite-dimensional inner product space, it is unknown whether $\mathcal{F}(E)$ possesses a finitely additive state. Let us recall that this is not known (Dvurečenskij, 1992; Pták, 1988) even if E is a real, complex, or quaternion incomplete inner product space.

In addition, in Dvurečenskij and Pták (to appear), it is shown that, for every real, complex, or quaternion incomplete inner product space E, $\mathcal{F}(E)$ possesses no finitely additive state s such that $|\operatorname{Ran}(s)| \leq \aleph_0$. It is proved that if such an $\mathcal{F}(E)$ possesses a state s, then $\operatorname{Ran}(s) = [0, 1]$.

3. STATES WITH THE GLEASON PROPERTY

In the present section, we introduce the notion the Gleason property of an infinite-dimensional inner product spaces, and we give the main result of the paper, Theorem 3.7.

Let E be an inner product space. We say that E has the *Gleason property* if, for any finite-dimensional subspace E_n, of E, $n = \dim E_n \geq 3$, and for any finitely additive state s on $\mathcal{F}(E_n)$, we have $\mathrm{Ran}_1(s) = [\lambda, \mu]$, where $0 \le \lambda \le \mu \le 1$ are real numbers. For example, any real, complex, or quaternion inner product space of dimension at least three has the Gleason property, see Dvurečenskij and Pták (2002). In Section 4, we show that in Keller spaces it fails to hold.

Lemma 3.1. *Let E be an n-dimensional inner product space, $n \ge 3$, with the Gleason property*

(i) *If s is a finitely additive state on $\mathcal{F}(E)$, then $1/n \in \mathrm{Ran}_1(s)$.*
(ii) *A finitely additive state s takes only finitely-many values if and only if s is defined by (2.4).*

Proof: (i) Let $\mathrm{Ran}_1(s) = [\lambda, \mu]$. If $1/n < \lambda$, take an orthogonal basis $x_1, \ldots, x_n$ in E. Then $1 = \sum_{i=1}^n 1/n < \sum_{i=1}^n s(\mathrm{sp}(x_i)) = 1$, which is a contradiction. In a similar way we can show that $1/n \le \mu$.

(i) It is clear that the state (2.4) takes only finitely many values. Conversely, let s take only finitely many values. Then $\mathrm{Ran}_1(s)$ is a finite set, therefore, by (i), $\mathrm{Ran}_1(s) = \{1/n\}$ which proves that s is equal to the state defined by (2.4). □

We say that a system $\{x_i\}$ of nonzero mutually orthogonal elements of a subspace M of E is *maximal* in M, MOS for abbreviation, if, for $x \in M$ such that $x \perp x_i$ for any i, we have $x = 0$.

Lemma 3.2. *Let $\{y_i\}$ be an MOS in a splitting subspace M of E. Then $\{y_i\}^{\perp\perp} = M$.*

Proof: It is evident that $\{y_i\}^{\perp\perp} \subseteq M$, so that $M^\perp \subseteq \{y_i\}^\perp$. On the other hand, let $x \in \{y_i\}^\perp$, then $x = x_1 + x_2$, where $x_1 \in M$ and $x_2 \in M^\perp$. Since $(x, y_i) = 0$ for any i, we have $0 = (x_1, y_i) + (x_2, y_i) = (x_1, y_i)$. The maximality of $\{y_i\}$ in M gives $x_1 = 0$, so that $x \in M^\perp$, which means $\{y_i\} \subseteq M^\perp$. □

A mapping $\phi : \mathcal{F}(E_1) \to \mathcal{F}(E_2)$ is said to be a *homomorphism* if $\phi(E_1) = E_2$, $\phi(M^\perp) = \phi(M)^\perp$, and $\phi(M \vee N) = \phi(M) \vee \phi(N)$ for all $M \perp N$. If ϕ is injective and ϕ^{-1} is a homomorphism, ϕ is said to be an *isomorphism*.

Proposition 3.3. *Let E_1 and E_2 be two inner product spaces over the same field, and let* $\dim E_1 = \dim E_2 < \infty$. *Then $\mathcal{F}(E_1)$ and $\mathcal{F}(E_2)$ are isomorphic.*

Proof: Let $\dim E_1 = n$ and let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be orthogonal bases in E_1 and E_2, respectively. We define a mapping $U : E_1 \to E_2$ by $U(x) = \sum_{i=1}^n \alpha_i y_i$ whenever $x = \sum_{i=1}^n \alpha_i x_i$. Set $\phi(M) := \{U(x) \colon x \in M\}$, $M \in \mathcal{F}(E_1)$. Then ϕ is an isomorphism of $\mathcal{F}(E_1)$ and $\mathcal{F}(E_2)$. □

Lemma 3.4. *Let $n \geq 1$ be an integer and $\{x_i^1\} \cup \{x_i^2\} \cup \cdots \cup \{x_i^n\} (i \in I)$ be a MOS of an infinite-dimensional inner product space E. Let E_n be an n-dimensional subspace of E. Then there is a mapping ϕ_n: $\mathcal{F}(E_n) \to \mathcal{F}(E)$ with the following properties:* (i) *$\phi_n(E_n) = E$,* (ii) *if $M, N \in \mathcal{F}(E_n)$, then $M \perp N$ if and only if $\phi_n(M) \perp (M)$ and, moreover, $\phi_n(M \vee N) = \phi_n(M) \vee \phi_n(M)$, and* (iii) ϕ_n $(\mathrm{sp}(x_j)) = \bigvee_i \mathrm{sp}(x_i^j)$, *$j = 1, \ldots, n$, where $x_1, \ldots, x_n$ is a fixed orthogonal basis in E_n. If s is a finitely additive state on $\mathcal{F}(E)$, then $s_n := s \circ \phi_n$ is a state on $\mathcal{F}(E_n)$.*

Proof: Let $x_1, \ldots, x_n$ be a fixed orthogonal basis in E_n and let $E_{\beta_i^n}$ be an n-dimensional subspace of E generated by $\{x_i^1, \ldots, x_i^n\} (i \in I)$. Let $\phi_{\beta_i^n}$ be an isomorphism from $\mathcal{F}(E_n)$ onto $\mathcal{F}(E_{\beta_i^n})$ such that $\phi_{\beta_i^n}(\mathrm{sp}(x_j)) = \mathrm{sp}(x_i^j)$ for $j = 1, \ldots, n$. Define a mapping $\phi_n : \mathcal{F}(E_n) \to \mathcal{F}(E)$, by setting

$$\phi_n(M) := \bigvee_i \phi_{\beta_i^n}(M), \quad M \in \mathcal{F}(E_n), \tag{3.1}$$

where $\bigvee$ means the join taken in the complete lattice $\mathcal{F}(E)$. Then ϕ_n is a map with the required properties. Indeed, $\phi_n(E_n) = \bigvee \phi_{\beta_i^n}(E_n) = \bigvee_i S_{\beta_i^n} = (\bigcup_i S_{\beta_i^n})^{\perp\perp} = S$. Further, if $M \perp N$, $M, N \in \mathcal{F}(E_n)$, then $\phi_{\beta_i^n}(M) \perp \phi_{\beta_j^n}(N)$ for all $i, j \in I$, and $\phi_{\beta_i^n}(M) \subseteq \phi_{\beta_j^n}(N)^\perp$, i.e., $\phi(M) \subseteq \phi_{\beta_j^n}(N)^\perp$. This yields $\phi(M)^\perp \supseteq \bigvee_j \phi_{\beta_j^n}(N) = \phi(N)$. Conversely, $\phi(M) \perp \phi(N)$ implies $\phi_{\beta_i^n}(M) \subseteq \bigvee_i \phi_{\beta_i^n}(M) \subseteq \bigcap_i \phi_{\beta_i^n}(N)^\perp \subseteq \phi_{\beta_i^n}(N)^\perp$. Therefore, $\phi_{\beta_j^n}(M) \perp \phi_{\beta_j^n}(N)$ for any i, which gives $M \perp N$. In addition, $\phi(M \vee N) = \bigvee_i \phi_{\beta_i^n}(M \vee N) = \bigvee_i (\phi_{\beta_i^n}(M) \vee_{\beta_i^n} \phi_{\beta_i^n}(N)) = \bigvee_i (\phi_{\beta_i^n}(M) \vee_{\beta_i^n} \phi_{\beta_i^n}$ $(N)) = \phi(M) \vee \phi(N)$, where $\vee_{\beta_i^n}$ denotes the join taken in the space $\mathcal{F}(E_{\beta_i^n})$. Indeed, since $E_{\beta_i^n}$ is finite-dimensional, the join $\vee_{\beta_i^n}$ is equal to the span generated by the subspaces $\phi_{\beta_i^n}(M)$ and $\phi_{\beta_i^n}(N)$, and it is equal to the join taken in the space $\mathcal{F}(E)$. $\square$

Lemma 3.5. *Let E be an infinite-dimensional inner product space with the Gleason property. Then there is no finitely additive state on $\mathcal{F}(E)$ with only finitely many values.*

Proof: Let $n \geq 3$ and let ϕ_n be the mapping from $\mathcal{F}(E_n)$ into $\mathcal{F}(E)$ guaranteed by Leema 3.4. Let s be a finitely additive state on $\mathcal{F}(E)$. Then the mapping s_n on $\mathcal{F}(E_n)$ defined by $s_n(M) = s(\phi_n(M))$, $M \in \mathcal{F}(E_n)$ is a state on $\mathcal{F}(E_n)$. If s takes only finitely many values on $\mathcal{F}(E)$, then $\mathrm{Ran}(s_n)$ is finite. Due to (ii) of Lemma 3.1, $\mathrm{Ran}(s_n) = \{0, 1/n, 2/n, \ldots, n/n\} \supseteq \mathrm{Ran}(s)$ for any $n \geq 3$. Hence, $\mathrm{Ran}(s)$ cannot be finite. $\square$

Lemma 3.6. *If E is an infinite-dimensional inner product space with the Gleason property, then $\mathcal{F}(E)$ possesses no finitely additive state with only countably many values.*

Proof: Suppose that s is a finitely additive state on $\mathcal{F}(E)$ such that $|\mathrm{Ran}(s)| = \aleph_0$. Then $\mathrm{Ran}(s)$ has to contain all rational numbers from the interval [0, 1]. Indeed, let us take the state s_n on $\mathcal{F}(E_n)$ from the proof of Lemma 3.4. Then every s_n has only finitely many values, i.e., $\mathrm{Ran}(s_n) = \{0, 1/n, 2/n, \ldots, n/n\} \subseteq \mathrm{Ran}(s)$ for any $n \geq 3$. It follows that $\mathrm{Ran}(s)$ contains all rational numbers from [0, 1].

Let now $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$ be a MOS in S such that $|I| = |J|$. Since J is infinite, express J in two forms $J = J_1 \cup J_2 = J_1' \cup J_2' \cup J_3'$, where $J_1 \cap J_2 = \emptyset = J_1' \cap J_2' = J_1' \cap J_3' = J_2' \cap J_3'$ and $|I| = |J_1| = |J_2| = |J_1'| = |J_2'| = |J_3'|$.

Without any loss of generality, we can assume in the first case that our MOS is of the form $\{x_i\} \cup \{u_i\} \cup \{\upsilon_i\}(i \in I)$, and in the second case is of the form $\{x_i\} \cup \{a_i\} \cup \{b_i\} \cup \{c_i\}(i \in I)$.

Set $M_0 = \bigvee_i \mathrm{sp}(x_i) \in \mathcal{F}(E)$.

In the first case, choose a three-dimensional subspace E_3 with a fixed orthogonal basis $\{x, u, v\}$ and apply Leema 3.5 to out case to obtain the embedding $\phi_3 : \mathcal{F}(E_3) \to \mathcal{F}(E)$ satisfying (i)–(iii) of the leema. Since s is a finitely additive state, the mapping s_3 on $\mathcal{F}(E_3)$ defined by $s_3(M) = s(\phi_3(M))$, $M \in \mathcal{F}(E_3)$, is a finitely additive state. We have $s_3(\mathrm{sp}(x)) = s(M_0)$, which due to (ii) of Leema 3.1 implies that $s(M_0) = 1/3$.

In the second case, choose a four-dimensional subspace E_4 with a fixed orthogonal basis $\{x, a, b, c\}$ and apply Leema 3.4 to our case to obtain the embedding $\phi_4 : \mathcal{F}(E_4) \to \mathcal{F}(E)$ satisfying (i)–(iii) of the leema. Since s is a finitely additive state, the mapping s_4 on $\mathcal{F}(E_4)$ defined by $s_4(M) = s(\phi_4(M))$, $M \in \mathcal{F}(E_4)$, is a finitely additive state. We have $s_4(\mathrm{sp}(x)) = s(M_0)$, which by the hypothesis implies that $s(M_0) = 1/4$.

Comparing both cases, we have $1/3 = s(M_0) = 1/4$. This is contradiction. □

We now present the main result of the paper.

Theorem 3.7. *Let $(E, K, (\cdot, \cdot))$ be an infinite-dimensional inner product space with the Gleason property. If s is a finitely additive state on $\mathcal{F}(E)$, then* $\mathrm{Ran}(S) = [0, 1]$.

Proof: Let s be a finitely additive state on $\mathcal{F}(E)$. If $\{x_i\}$ is a MOS in $M \in \mathcal{F}(E)$, then for $M_0 = \bigvee_i \mathrm{sp}(x_i)$ we have $M_0 \subseteq M$ and $s(M_0) = s(M)$. Indeed, since $M_0 = M_0 \vee (M \cap M_0^\perp)$, we see that $s(M_0) = s(M_0) + s(M \cap M_0^\perp) = s(M_0) + 1 - s(M^\perp \vee M_0) = s(M)$.

Since $|\mathrm{Ran}(s)| > \aleph_0$ (Lemma 3.5), there is a MOS $\{x_i\} \cup \{u_i\}(i \in I)$ in S such that, for $M = \bigvee_i \mathrm{sp}(x_i)$, $\alpha := s(M)$ irrational. Without any loss of generality, we can assume $\alpha > 1/2$ (if it is not the case, we pass to $M^\perp$).

For any $n \geq 3$, we express $\{u_i\}$ as a join of mutually disjoint sets $\{u_i^1\} \cup \cdots \cup \{u_i^{n-1}\}(i \in I)$. Applying Lemma 3.4, there is a mapping ϕ_n: $\mathcal{F}(E_n) \to \mathcal{F}(E)$ satisfying (i)–(iii) of the lemma. Then $\phi_n(\mathrm{sp}(x)) = M_0$ and the mapping s_n on $\mathcal{F}(E_n)$

defined by $s_n(N) = s(\phi_n(N))$, $N \in \mathcal{F}(E_n)$, is a state on $\mathcal{F}(E_n)$, and, moreover, $s_n(\mathrm{sp}(x)) = s(M_0) = \alpha$. By (i) of Lemma 3.1, $\alpha \in \mathrm{Ran}(s_n)_1 = [\lambda_n, \mu_n]$, where $0 \le \lambda_n \le \mu_n \le 1$. Therefore, $\mathrm{Ran}(s) \supseteq \mathrm{Ran}(s_n)_1 \supseteq [1/n, \alpha]$, while $\lambda_n \le 1/n$. Hence, $\mathrm{Ran}(s) \supseteq \bigcup_{n=1}^{\infty}[1/n, \alpha] = (0, \alpha]$, so that $\mathrm{Ran}(s) \supseteq [0, \alpha]$. Since $\alpha > 1/2$, it is sufficient to consider the complements to obtain $\mathrm{Ran}(s) = [0, 1]$. □

As a corollary we have the following result from Dvurečenskij and Pták (2002) characterizing finitely additive states on $\mathcal{F}(S)$ for a pre-Hilbert space S.

Theorem 3.8. *If s is a finitely additive state on $\mathcal{F}(S)$ of an infinite-dimensional real, complex or quaternion inner product space S, then* $\mathrm{Ran}(S) = [0, 1]$.

Proof: From the Gleason theorem we have that S satisfies the Gleason property (see Dvurečenskij and Pták, 2002). □

Let us recall that Theorem 3.8 does not hold for states on $\mathcal{E}(S)$, in general, In fact, it is shown in Pták and Weber (2001) that there is an infinite-dimensional pre-Hilbert space S such that $\mathcal{E}(S)$ possesses a two-values state (see also Chetcuti, 2002, for further analysis).

4. GLEASON PROPERTY AND KELLER SPACES

One outstanding problem of quantum logic theory is that of the characterization of quantum logics to be isomorphic with the set $\mathcal{L}(H)$ of all closed subspaces of a separable complex or real or quaternion Hilbert space H. Many specialists have thought that the properties as atomicity, exchange axiom, infinite-dimensionality, irreducibility of a complete OML L are characteristics only of $\mathcal{L}(H)$; see e.g. Varadarajan (1968). Therefore, a result by Keller (1980) was a great surprise for quantum logicians, when he presented OMLs with all the above properties which cannot be embedded into $\mathcal{L}(H)$ for any H.

For a detailed theory of Keller spaces and measure theory on them see Keller (1980, 1988, 1990) or Dvurečenskij (1992, Section 5.4).

In what follows, we show that in any infinite-dimensional Keller space E of type $(n_0, n_1, n_2, \ldots)$, where $n_k \ge 3$ for any $k = 0, 1, \ldots$, there is a sequence $\{s_k\}$ of finitely additive states on $\mathcal{F}(E)$ such that $\mathrm{Ran}(s_k) = \{0, 1/n_k, 2/n_k, \ldots, n_k/n_k$ for $k \ge 0$.

Theorem 4.1. *In every infinite-dimensional Keller space E of type $(n_0, n_1, n_2, \ldots)$, where $n_k \ge 3$ for any $k = 0, 1, \ldots$, there is a σ-additive state s_k on $\mathcal{F}(E)$ such that Ran$(s_k) = \{0, 1/n_k, 2/n_k, \ldots, n_k/n_k$ for $k = 0, 1, \ldots$. In particular, in such an E the Gleason property fails to hold.*

Proof: Let E be a Keller infinite-dimensional inner product space of type $(n_0, n_1, n_2, \ldots)$, where $n_k \geq 3$ for any $k = 0, 1, \ldots$. There is an orthogonal basis $\{e_1, \ldots, e_{n_0}\} \cup \{e_{n_0+1}, \ldots, e_{n_0+n_1}\} \cup \{e_{n_0+n_1+1}, \ldots e_{n_0+n_1+n_2}\} \cup \cdots$. Let S_k be the subspace of E generated by $\{e_i : \sum_{j=0}^{k-1} n_j < i \leq \sum_{j=0}^{k} n_j\}$. There is a homomorphism $\prod_k : \mathcal{F}(E) \to \mathcal{L}(\mathbb{R}^{n_k})$, $k = 0, 1, \ldots$, which preserves all atoms from $\mathcal{F}(S_k)$ and vanishes on all atoms from $\mathcal{F}(S_j)$ for $j \neq k$. Therefore, if s_k is a state on $\mathcal{L}(\mathbb{R}^{n_k})$, then

$$\widehat{s_k}(M) := s_k(\prod_k(M)), \quad M \in \mathcal{F}(E),$$

is a σ-additive state on $\mathcal{F}(E)$ (for details see, Dvurečenskij, 1992, Section 5.4.6).

In particular, if we set $s_k(M) := \dim(\prod_k(M))/n_k$, $M \in \mathcal{F}(E)$, then $\mathrm{Ran}(s_k) = \{0, 1/n_k, 2/n_k, \ldots, n_k/n_k\}$ for $k = 0, 1, \ldots$. In particular, the Gleason property fails to hold in this Keller space. $\square$

We now present another example of an infinite-dimensional inner product space in which the Gleason property fails.

Example 4.2. Let $\mathbb{Q}$ be the set of all rational numbers. Denote by $\mathbb{Q}^f$ the set of all infinite sequences $q = (q_1, q_2, \ldots)$ from $\mathbb{Q}^f$ such that all coordinates of $(q_1, q_2, \ldots)$ are zero unless finitely many of them. Then $\mathbb{Q}^f$ is an infinite-dimensional vector space over the field $\mathbb{Q}$ with the involution $\lambda \longmapsto \lambda$, $\lambda \in \mathbb{Q}$. The bilinear form $\langle q, p\rangle = \sum_{i=1}^{\infty} q_i p_i$, where $q = (q_1, q_2, \ldots)$, $p = (p_1, p_2, \ldots) \in \mathbb{Q}^f$ is a Hermitian one, and $(\mathbb{Q}^f, \mathbb{Q}, \langle\cdot,\cdot\rangle)$ is an infinite-dimensional inner product space.

Let E_n be any finite-dimensional subspace of $\mathbb{Q}^f$, $\dim E_n \geq 3$, and let x be any nonzero vector in E_n. The mapping

$s_x : \mathcal{F}(E_n) \to [0, 1]$ defined via

$$s_x(M) = \frac{\langle x_M, x_M\rangle}{\langle x, x\rangle}, \quad M \in \mathcal{F}(E_n),$$

where $x = x_M + x_{M^\perp}$ and $x_M \in M$, $x_{M^\perp} \in E_n \cap M^\perp$, is a finitely additive state on $\mathcal{F}(E_n)$ concentrated on $\mathrm{sp}(x)$. In particular, we have

$$s_x(sp(f)) = \frac{\langle f, x\rangle^2}{\langle f, f\rangle\langle x, x\rangle}$$

for any nonzero $f \in E_n$. Then $0, 1 \in \mathrm{Ran}_1(s_x)$ and $\mathrm{Ran}_1(s_x)$ takes only rational values, so that it cannot be a closed interval $[\lambda, \mu]$ for some $1 \leq \lambda \leq \mu \leq 1$. Hence, $\mathbb{Q}^f$ does not satisfy the Gleason property.

We recall that we do not know whether there is a finitely additive state on $\mathcal{F}(\mathbb{Q}^f)$.

Let us note that it would be interesting to know whether there are also further infinite-dimensional inner product spaces other as pre-Hilbert spaces which have the Gleason property.

ACKNOWLEDGMENT

This work has been supported by the grant 2/3163/23 SAV, Bratislava, Slovakia.

REFERENCES

Armstrong, T. E. and Prikry, K. (1981). Liapounoff's theorem for nonatomic, finitely-additive, bounded, finite-dimensional, vector-valued measures. *Transactions of American Mathematical Society*, **266**, 499–514.

Chetcuti, E. (2002). *Completeness Criteria for Inner Product Spaces*, MSc Thesis, University of Malta.

Dvurečenskij, A. (1992). *Gleason's Theorem and Its Applications*, Kluwer, Dordrecht.

Dvurečenskij, A. and Pták, P. (2002). On states on orthogonally closed subspaces of an inner product space. *Letters Mathematical Physics*, **62**, 63–70.

Gleason, A. M. (1957). Measures on the closed subspaces of a Hilbert space. *Journal of Mathematical Mechanics*, **6**, 885–893.

Keller, H. A. (1980). Ein nicht-klassischer Hilbertischen Raum. *Mathematics Zeitschrift*, **172**, 41–49.

Keller, H. A. (1988). Measures on orthomodular vector space lattices. *Studies in Mathemattics*, **88**, 183–195.

Keller, H. A. (1990). Measures on infinite-dimensional orthomodular spaces. *Foundations of Physics*, **20**, 575–604.

Maeda, F. and Maeda, S. (1970). *Theory of Symmetric Lattices*, Springer-Verlag, Berlin.

Pták, P. (1988). FAT $\leftrightarrow$ CAT (in the state space of quantum logics). In *Proceedings of the First Winter School on Measure Theory, Liptovský Ján, Jan.* 10–15, 1988, A. Dvurečenskij and S. Pulmannová, eds. pp. 113–118.

Pták, P. and Weber, H. (2001). Lattice properties of closed subspaces of inner product spaces. *Proceedings of the American Mathematical Society*, **129**, 2111–2117.

Varadarajan, V. S. (1968). *Geometry of Quantum Theory, Vol 1*, Van Nostrand, Princeton, NJ.